Advanced Microeconomics: Lecture Note 7

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1 Multiple levels of performance

1 We consider a production process where n possible outcomes can be realized:

$$q_1 < q_2 < \cdots < q_n.$$

We denote the principal's return in each of those outcomes by $S_i = S(q_i)$.

2 Let $\lambda_{ik} > 0$ be the probability that production q_i takes places when the effort level is e_k. Denote $\Delta\lambda_i = \lambda_{i1} - \lambda_{i0}$.

3 A contract is an n-tuple of payments $(t_1, t_2, \ldots, t_n)$.

1.1 Limited liability

4 Suppose $l = 0$.

5 To induce $e = 1$, the principal's problem is:

$$\begin{aligned}
\underset{t_1,\ldots,t_n}{\text{maximize}} \quad & \sum_{i=1}^{n} \lambda_{i1}(S_i - t_i), \\
\text{subject to} \quad & \sum_{i=1}^{n} \lambda_{i1} t_i - \psi \geq 0, \\
& \sum_{i=1}^{n} \lambda_{i1} t_i - \psi \geq \sum_{i=1}^{n} \lambda_{i0} t_i, \\
& t_i \geq 0.
\end{aligned}$$

6 IR constraint is implied by IC constraint and LL constraints:

$$\sum_{i=1}^{n} \lambda_{i1}(S_i - t_i) \geq \sum_{i=1}^{n} \lambda_{i0} t_i \geq 0.$$

7 The Lagrangian is

$$\mathcal{L}(t_1, \ldots, t_n, \gamma, \mu_1, \ldots, \mu_n) = \sum_{i=1}^{n} \lambda_{i1}(S_i - t_i) + \gamma \left[\sum_{i=1}^{n} \lambda_{i1} t_i - \psi - \sum_{i=1}^{n} \lambda_{i0} t_i \right] + \sum_{i=1}^{n} \mu_i t_i.$$

8 The first-order condition on each t_i is

$$-\lambda_{i1} + \gamma(\lambda_{i1} - \lambda_{i0}) + \mu_i \leq 0,$$

with the equality when $t_i^{\text{SB}} > 0$ and the slackness conditions $\mu_i t_i^{\text{SB}} = 0$.

9 There are some second-best transfers t_i^{SB} to be strictly positive. Otherwise, IR constraint cannot be satisfied.

10 For i such that $t_i^{\text{SB}} > 0$, we have $\mu_i = 0$ by the slackness condition.

Thus, the first-order condition implies

$$\gamma = \frac{\lambda_{i1}}{\lambda_{i1} - \lambda_{i0}}.$$

11 If there are $i \neq j$ such that $t_i^{\text{SB}} > 0$ and $t_j^{\text{SB}} > 0$ (other $t_k^{\text{SB}} = 0$), then

$$\frac{\lambda_{i1}}{\lambda_{i1} - \lambda_{i0}} = \gamma = \frac{\lambda_{j1}}{\lambda_{j1} - \lambda_{j0}}.$$

On the other hand, if $\frac{\lambda_{i1}}{\lambda_{i1} - \lambda_{i0}} > \frac{\lambda_{j1}}{\lambda_{j1} - \lambda_{j0}}$, then it should be the case that $\gamma = \frac{\lambda_{j1}}{\lambda_{j1} - \lambda_{j0}}$, $\mu_i > 0$, and $t_i^{\text{SB}} = 0$.

12 If the ratios $\frac{\lambda_{i1} - \lambda_{i0}}{\lambda_{i1}}$ are all different, there exists a single index j such that $\frac{\lambda_{j1} - \lambda_{j0}}{\lambda_{j1}}$ is the highest ratio.

Then we should have $t_j^{\text{SB}} > 0$ and $t_k^{\text{SB}} = 0$ ($k \neq j$). That is, the structure of the optimal payments is bang-bang.

The agent receives a strictly positive transfer only in this particular outcome j, and this payment is such that IC constraint is binding, $t_j^{\text{SB}} = \frac{\psi}{\lambda_{j1} - \lambda_{j0}}$.

In all other outcomes, the agent receives no transfer and $t_k^{\text{SB}} = 0$ for all $k \neq j$.

Finally, the agent gets a strictly positive ex ante limited liability rent that is worth $\lambda_{j1} \frac{\psi}{\lambda_{j1} - \lambda_{j0}} - \psi = \frac{\lambda_{j0}\psi}{\lambda_{j1} - \lambda_{j0}}$.

13 The agent is rewarded in the outcome that is the most informative about the fact that he has exerted a positive effort.

Indeed, $\frac{\lambda_{i1} - \lambda_{i0}}{\lambda_{i1}}$ can be interpreted as a likelihood ratio.

The principal therefore uses a maximum likelihood ratio criterion to reward the agent. The agent is only rewarded when this likelihood ratio is maximum.

Like an econometrician, the principal tries to infer from the observed output what has been the parameter (effort) underlying this distribution. But here the parameter is endogenous and affected by the incentive contract.

14 The probabilities of success is said to satisfy the monotone likelihood ratio property (MLPR) if $\frac{\lambda_{i1} - \lambda_{i0}}{\lambda_{i1}} = 1 - \frac{\lambda_{i0}}{\lambda_{i1}}$ is increasing in i.

If $i = 2$ (there are only two outcomes), then this property reduces to $\lambda_1 > \lambda_0$ as previously.

15 Theorem: If the probability of success satisfies MLRP, the second-best payment t_i^{SB} received by the agent may be chosen to be increasing with the level of production q_i.

(a) Let J be the set of indices j such that $\frac{\lambda_{j1} - \lambda_{j0}}{\lambda_{j1}} = \max_i \{ \frac{\lambda_{i1} - \lambda_{i0}}{\lambda_{i1}} \}$.

(b) If $J = \{n\}$, then we have $t_n^{\text{SB}} = \frac{\psi}{\lambda_{n1} - \lambda_{n0}}$ and $t_i^{\text{SB}} = 0$ for $i < n$.

(c) Otherwise, we have $t_i^{\text{SB}} = 0$ if $i \notin J$. For $i \in J$, the transfer t_i^{SB} should make IC binding. Thus,

$$\sum_{i \in J} (\lambda_{i1} - \lambda_{i0}) t_i^{\text{SB}} = \psi.$$

The principal (and the agent) are indifferent to the profiles of positive transfers.

- Principal: $\sum_{i=1}^n \lambda_{i1}(S_i - t_i^{\text{SB}}) = \sum_{i=1}^n \lambda_{i1} S_i - \sum_{i\in J} \lambda_{i1} t_i^{\text{SB}} = \sum_{i=1}^n \lambda_{i1} S_i - \sum_{i\in J} \gamma(\lambda_{i1} - \lambda_{i0}) t_i^{\text{SB}} = \sum_{i=1}^n \lambda_{i1} S_i - \gamma\psi$ is fixed.
- Agent: $\sum_{i=1}^n \lambda_{i1} t_i^{\text{SB}} - \psi = \sum_{i\in J} \lambda_{i1} t_i^{\text{SB}} - \psi = \sum_{i\in J} \gamma(\lambda_{i1} - \lambda_{i0}) t_i^{\text{SB}} - \psi = \gamma\psi - \psi$ is also fixed.

For example, they can be chosen positive and increasing.

16 MLRP is stronger than the first-order stochastic dominance.

1.2 Risk aversion

17 Suppose now that the agent is strictly risk-averse. The optimal contract that induces effort must solve the program below:

18 To induce $e = 1$, the principal's problem is:

$$
\begin{aligned}
\underset{t_1,\ldots,t_n}{\text{maximize}} \quad & \sum_{i=1}^n \lambda_{i1}(S_i - t_i), \\
\text{subject to} \quad & \sum_{i=1}^n \lambda_{i1} u(t_i) - \psi \geq 0, \\
& \sum_{i=1}^n \lambda_{i1} u(t_i) - \psi \geq \sum_{i=1}^n \lambda_{i0} u(t_i).
\end{aligned}
$$

19 Let $u_i = u(t_i)$. Then the principal's program can be written as:

$$
\begin{aligned}
\underset{u_1,\ldots,u_n}{\text{maximize}} \quad & \sum_{i=1}^n \lambda_{i1}(S_i - u^{-1}(u_i)), \\
\text{subject to} \quad & \sum_{i=1}^n \lambda_{i1} u_i - \psi \geq 0, \\
& \sum_{i=1}^n \lambda_{i1} u_i - \psi \geq \sum_{i=1}^n \lambda_{i0} u_i.
\end{aligned}
$$

20 The Lagrangian is

$$
\mathcal{L} = \sum_{i=1}^n \lambda_{i1}(S_i - u^{-1}(u_i)) + \mu\left[\sum_{i=1}^n \lambda_{i1} u_i - \psi\right] + \gamma\left[\sum_{i=1}^n \lambda_{i1} u_i - \psi - \sum_{i=1}^n \lambda_{i0} u_i\right].
$$

21 The first-order condition is for each i,

$$
\frac{1}{u'(t_i^{\text{SB}})} = \mu + \gamma\left[1 - \frac{\lambda_{i0}}{\lambda_{i1}}\right].
$$

22 Multiplying each of these equations by λ_{i1} and summing over i yields

$$
\mathbf{E}_q\left[\frac{1}{u'(t_i^{\text{SB}})}\right] = \sum_i \lambda_{i1} \frac{1}{u'(t_i^{\text{SB}})} = \mu,
$$

which is positive, where $\mathbf{E}_q(\cdot)$ denotes the expectation operator with respect to the distribution of outputs induced by effort $e = 1$.

23 Multiplying each of these equations by $\lambda_{i1}u(t_i^{\mathrm{SB}})$, summing over i, and taking into account the expression of μ obtained above yields

$$\begin{aligned}
\gamma\Big[\sum_{i=1}^{n}(\lambda_{i1}-\lambda_{i0})u(t_i^{\mathrm{SB}})\Big] &= \sum_{i=1}^{n}\lambda_{i1}u(t_i^{\mathrm{SB}})\Big[\frac{1}{u'(t_i^{\mathrm{SB}})}-\mu\Big] \\
&= \mathbf{E}_q\Big[u(t_i^{\mathrm{SB}})\Big(\frac{1}{u'(t_i^{\mathrm{SB}})}-\mu\Big)\Big] \\
&= \mathbf{E}_q\Big[u(t_i^{\mathrm{SB}})\Big(\frac{1}{u'(t_i^{\mathrm{SB}})}-\mathbf{E}_q\Big[\frac{1}{u'(t_i^{\mathrm{SB}})}\Big]\Big)\Big].
\end{aligned}$$

24 Using the slackness condition $\gamma\big[\sum_{i=1}^{n}\lambda_{i1}u_i-\psi-\sum_{i=1}^{n}\lambda_{i0}u_i\big]=0$ to simplify LHS:

$$\gamma\psi = cov\Big(u(t_i^{\mathrm{SB}}), u'(t_i^{\mathrm{SB}})\Big).$$

25 By assumption, u and u' covary in opposite directions. Moreover, a constant wage $t_i^{\mathrm{SB}}=t^{\mathrm{SB}}$ for all i does not satisfy the IC constraint, and thus t^{SB} cannot be constant everywhere.

Hence, RHS is necessarily strictly positive. Thus we have $\gamma>0$, and the IC constraint is binding.

26 For t_i^{SB} to be increasing with i, MLRP must again hold. Then higher outputs are also those that are the more informative ones about the realization of a high effort. Hence, the agent should be more rewarded as output increases.

2 A continuum of performances

27 We assume that outcomes q is drawn from a distribution $F(\cdot\mid e)$ on the support $[\underline{q},\bar{q}]$.

This distribution is conditional on the agent's effort $e\in\{0,1\}$. We denote by $f(\cdot\mid e)$ the density corresponding to the distribution $F(\cdot\mid e)$.

28 Complete information:

To induce $e=1$, the principal's problem is

$$\begin{aligned}
\underset{t(q)}{\text{maximize}}\quad & \int[S(q)-t(q)]f(q\mid 1)\,\mathrm{d}q,\\
\text{subject to}\quad & \int u(t(q))f(q\mid 1)\,\mathrm{d}q-\psi\geq 0.
\end{aligned}$$

Denoting the multipliers by γ. The Lagrangian is

$$\mathcal{L}(q,t)=(S(q)-t)f(q\mid 1)+\gamma\big[u(t)f(q\mid 1)-\psi\big].$$

Optimizing pointwise with respect to t yields

$$-f(q\mid 1)+\gamma u'(t)f(q\mid 1)=0.$$

Thus, $\gamma=\frac{1}{u'(t)}>0$ and the wage is constant. It implies that $t^*=u^{-1}(\psi)$, which is the same as the two-outcome case. The profit is

$$\int qf(q\mid 1)\,\mathrm{d}q-u^{-1}(\psi).$$

Had the principal decided to let the agent exert no effort, $e = 0$, he would (optimally) make a zero payment to the agent whatever the realization of profit. The payoff is $\int qf(q \mid 0)\,\mathrm{d}q$.

$e^* = 1$ is the optimal choice of principal if and only if

$$\int qf(q \mid 1)\,\mathrm{d}q - u^{-1}(\psi) \geq \int qf(q \mid 0)\,\mathrm{d}q.$$

29 In an environment with incomplete information, a contract $t(q)$ inducing a positive effort must satisfy the IC constraint

$$\int u(t(q))f(q \mid 1)\,\mathrm{d}q - \psi \geq \int u(t(q))f(q \mid 0)\,\mathrm{d}q,$$

and the IR constraint

$$\int u(t(q))f(q \mid 1)\,\mathrm{d}q - \psi \geq 0.$$

30 Incomplete information with a risk-neutral agent.

(1) To induce $e = 1$, the principal's problem is

$$\begin{aligned} \underset{t(q)}{\text{maximize}} \quad & \int [S(q) - t(q)]f(q \mid 1)\,\mathrm{d}q \\ \text{subject to} \quad & \int t(q)f(q \mid 1)\,\mathrm{d}q - \psi \geq 0 \\ & \int t(q)f(q \mid 1)\,\mathrm{d}q - \psi \geq \int t(q)f(q \mid 0)\,\mathrm{d}q \end{aligned}$$

Principal can set $t(q) = q - \int qf(q \mid 1)\,\mathrm{d}q + \psi$. The expected payoff is $\int qf(q \mid 1)\,\mathrm{d}q - \psi$.

(2) To induce $e = 0$, the principal's problem is

$$\begin{aligned} \underset{t(q)}{\text{maximize}} \quad & \int [S(q) - t(q)]f(q \mid 0)\,\mathrm{d}q \\ \text{subject to} \quad & \int t(q)f(q \mid 0)\,\mathrm{d}q \geq 0 \\ & \int t(q)f(q \mid 0)\,\mathrm{d}q \geq \int t(q)f(q \mid 1)\,\mathrm{d}q - \psi \end{aligned}$$

Principal can set $t(q) = 0$ or $t(q) = q - \int qf(q \mid 0)\,\mathrm{d}q$. The expected payoff is $\int qf(q \mid 0)\,\mathrm{d}q$.

(3) $e = 1$ is the optimal of principal if and only if

$$\int qf(q \mid 1)\,\mathrm{d}q - \psi \geq \int qf(q \mid 0)\,\mathrm{d}q.$$

31 Incomplete information with a risk-averse agent.

(1) To induce $e = 1$, the principal's problem is

$$\begin{aligned} \underset{t(q)}{\text{maximize}} \quad & \int [S(q) - t(q)]f(q \mid 1)\,\mathrm{d}q, \\ \text{subject to} \quad & \int u(t(q))f(q \mid 1)\,\mathrm{d}q - \psi \geq 0, \\ & \int u(t(q))f(q \mid 1)\,\mathrm{d}q - \psi \geq \int u(t(q))f(q \mid 0)\,\mathrm{d}q. \end{aligned}$$

Denoting the multipliers by γ and μ, respectively, the Lagrangian writes as

$$[S(q) - t]f(q \mid 1) + \gamma[u(t)[f(q \mid 1) - f(q \mid 0)] - \psi] + \mu[u(t)f(q \mid 1) - \psi].$$

Optimizing pointwise with respect to t yields

$$\frac{1}{u'(t^{\mathrm{SB}}(q))} = \mu + \gamma\left[1 - \frac{f(q \mid 1)}{f(q \mid 0)}\right].$$

We can verify that $\gamma > 0$ and $\mu > 0$. Then

$$u\left(\int t^{\mathrm{SB}}(q)f(q \mid 1)\,\mathrm{d}q\right) > \int u(t^{\mathrm{SB}}(q))f(q \mid 1)\,\mathrm{d}q = \psi.$$

That is, the expected wage $C^{\mathrm{SB}} = \int t^{\mathrm{SB}}(q)f(q \mid 1)\,\mathrm{d}q$ is larger than $u^{-1}(\psi) = C^*$.

(2) To induce $e = 0$, the principal's problem is

$$\begin{aligned}
\underset{t(q)}{\text{maximize}} \quad & \int [S(q) - t(q)]f(q \mid 0)\,\mathrm{d}q \\
\text{subject to} \quad & \int u(t(q))f(q \mid 0)\,\mathrm{d}q \geq 0 \\
& \int u(t(q))f(q \mid 0)\,\mathrm{d}q \geq \int u(t(q))f(q \mid 1)\,\mathrm{d}q - \psi
\end{aligned}$$

Principal can set $t(q) = 0$. The expected payoff is $\int qf(q \mid 0)\,\mathrm{d}q$.

(3) $e = 1$ is optimal if and only if

$$\int qf(q \mid 1)\,\mathrm{d}q - C^{\mathrm{SB}} \geq \int qf(\pi \mid 0)\,\mathrm{d}q.$$

32 In an optimal incentive scheme, compensation is not necessarily increasing in outcomes.

For optimal incentive scheme to be increasing, it must be that the likelihood ratio $\frac{f(q|e=0)}{f(q|e=1)}$ is decreasing in q. That is, as q increases, the likelihood of getting q if effort is $e = 1$ relative to the likelihood if effort is $e = 0$ must increase.

This property is known as monotone likelihood ratio property.