Advanced Microeconomics: Lecture Note 13

Instructor: Xiang Sun

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1 A decision maker faces a decision problem under uncertainty, whose payoff $U(a, \omega)$ depends on her action a and the state of the world $\omega \in \Omega$.

DM does not know ω but observes an informative random signal $s \in S$, drawn according to the information structure $\sigma \colon \Omega \to \Delta(S)$, which specifies the conditional probability $\sigma(s \mid \omega)$ of observing signal s when the state is ω.

The information structure σ can be viewed as an experiment P associated the set of outcomes S_P.

2 个体在做决策时，虽然其效用（payoff）与现实世界的真实状态（state）有关，但往往无法观察到真实的状态。为了估计真实的状态，个体会考虑进行试验（experiment）以获取一些能够反应真实状态的信号（signal）。试验的好坏可以用其提供的信息量（或者更高的期望效用）来衡量。Blackwell 定理为试验之间的比较提供了建议一个简单的刻画。

3 Let Ω be a finite set of states of nature, $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$. Let p be a vector of a priori probabilities associated with the states in Ω, $p = (p_1, p_2, \dots, p_n)$, where $\sum_{i=1}^n p_i = 1$, $p_i \geq 0$ for $i = 1, 2, \dots, n$.

4 Let P be an $n \times m$ row stochastic matrix. It represents an experiment with m outcomes $S_P = \{s_1, s_2, \dots, s_m\}$, where P_{ij} is the probability of outcome s_j when the state is ω_i.

Note that different experiments may involve different outcomes.

5 Let A be a finite set of actions that can be taken by the decision-maker (DM), $A = \{a_1, a_2, \dots, a_\ell\}$.

A payoff function $U \colon A \times \Omega \to \mathbb{R}$ associates payoffs to each action and state pair. The function U can be depicted by an $\ell \times n$ matrix, denoted U, the element $U_{ki} = U(a_k, \omega_i)$ of which is the payoff gained when an action a_k is taken and the state turns out to be ω_i.

6 The DM can only observe the outcomes, not the states, and chooses actions accordingly.

The DM's strategy is delineated by an $m \times \ell$ row stochastic matrix D, the element D_{jk}, of which determines the probability that the DM takes action a_k on observing signal s_j.

The DM wishes to optimize D to obtain the maximum expected payoff.

7 The action distribution under the experiment P and strategy D conditional on ω_i is

$$\left((PD)_{i1}, (PD)_{i2}, \dots, (PD)_{im}\right).$$

So the action distributions under the experiment P and strategy D is PD.

As D varies, the set of all action distribution is

$$C(P) = \{PD \mid D \text{ is a row stochastic matrix}\}.$$

8 The expected payoff under the experiment P and strategy D conditional on ω_i is

$$\sum_{j=1}^{m}\left[P_{ij}\sum_{k=1}^{\ell}\left(D_{jk}\cdot U_{ki}\right)\right],$$

which is

$$\sum_{j=1}^{m}\left[P_{ij}\cdot(DU)_{ji}\right]=(PDU)_{ii}.$$

So the expected payoff vector

$$\text{diag}(PDU)=\big((PDU)_{11},(PDU)_{22},\ldots,(PDU)_{nn}\big)\,.$$

As D varies, the set of all possible expected payoff vectors is

$$B(P,U)=\{\text{diag}(PDU)\mid D\text{ is a row stochastic matrix}\}.$$

9 Given the prior p on Ω, the expected payoff under the experiment P and strategy D is

$$\sum_{i=1}^{n}p_i\left[\sum_{j=1}^{m}\left[P_{ij}\sum_{k=1}^{l}\left(D_{jk}\cdot U_{ki}\right)\right]\right]=\text{Trace}(PDU\hat{p}),$$

where $\hat{p}$ be an $n\times n$ matrix containing the elements of p in its main diagonal and zero elsewhere.

10 Maximization of Trace$(PDU\hat{p})$ is obtained by solving a linear programming problem for the elements of D constrained by the properties of a row stochastic matrix.

Denote $F(P,U,p)=\max_D Trace(PDU\hat{p})$.

11 Let P and Q be two experiments operating on the same set of state Ω.

From the economic point of view, Q is defined to be generally more informative than P if the maximal expected payoff yielded by P is not larger than that yielded by Q for all payoff matrices U and all probability vectors p. Formally:

Q is more informative than P if $F(Q,U,p)\geq F(P,U,p)$ for all U and any p.

12 Blackwell's theorem: For any two experiments P (an $n\times m$ matrix) and Q (an $n\times m'$ matrix), the following are equivalent:

(a) $F(Q,U,p)\geq F(P,U,p)$ for any U and any p.

(b) For any U, $B(Q,U)\supseteq B(P,U)$.

(c) $C(Q)\supseteq C(P)$.

(d) There exists an $m\times m'$ row stochastic matrix M such that $P=QM$.

Here, the matrix M is called the garbling matrix.

The first ranking comes from thinking in terms of expected utility: Say that Q is more informative than P if every Bayesian agent, facing any decision problem, can obtain a higher expected utility using Q than by using P.

The third ranking comes from a notion of feasibility: We can then rank Q and P according to which yields the larger set of feasible conditional distributions of actions.

The last ranking comes from a notion of "adding noise": Say that P is a garbling of Q if DM who knows Q could replicate P by randomly drawing a signal $s \in S_P$ after each observation of $s \in S_Q$.

13 The proofs of "b $\rightarrow$ a", "c $\rightarrow$ b", and "d $\rightarrow$ c" are trivial.

14 The proof of "a $\rightarrow$ d":

(a) Suppose for every $m \times m'$ row stochastic matrix M, $Q \neq PM$.

(b) Then $Q \notin E$, where

$$E = \{PM \mid M \text{ is an } m \times m' \text{ row stochastic matrix}\}.$$

(c) Note that E is a closed convex set in $\mathbb{R}^{n \times m'}$.

(d) Notice that each linear functional on the space of $n \times m'$ matrices is in the form of Trace$(G^t \cdot)$.

(e) By hyperplane separating theorem, there exists an $n \times m'$ matrix G such that for any $m \times m'$ row stochastic matrix M,

$$\text{Trace}(G^t Q) > \text{Trace}(G^t PM).$$

(f) Let $U^t = \hat{p}^{-1} G$. Then

$$\text{Trace}(PDU\hat{p}) = \text{Trace}(PDG^t) = \text{Trace}(G^t PD) < \text{Trace}(G^t Q) = \text{Trace}(QU\hat{p}).$$

(g) Thus,

$$\max_D \text{Trace}(PDU\hat{p}) < \text{Trace}(QU\hat{p}) \leq \max_D \text{Trace}(QDU\hat{p}).$$

Contradiction.

15 简单来说，Blackwell 定理说明了，如果试验 P 比试验 Q 拥有的信息量更丰富，那么 $Q = PM$。这个矩阵 M 描述的是通过"篡改"试验 P 的结果来得到试验 Q 结果的过程，并且这一篡改过程与真实的状态毫无关系。由于矩阵 M 是一个行随机矩阵，所以通过试验 P 得到的后验概率（posterior）是通过试验 Q 得到的后验概率（posterior）的保留均值的伸展（mean-preserving spread），这一意味着前者承受的期望风险更小。

Task

- Reading:
- Understanding: